The Rock-Paper-Scissors Game with Bystanders

Carl T. Bergstrom & Benjamin Kerr

Here we provide some details on the simple model described in our News & Views piece "Taking the bad with the good."

The RPS game without bystanders

Consider an infinite population of individuals, each playing a pure strategy: Rock, Paper, or Scissors (denoted R, P or S, respectively). We let x_i be the proportion of the population playing strategy i, where $i \in \{R, P, S\}$. Individuals form pairs at random and play the game according to the standard rules. Payoffs are 1 for winning or drawing, and 0 for losing. To simplify notation, we use the functions $v(i)$ and $e(i)$ to represent the victim and enemy of strategy i (e.g., $v(R) = S = e(P)$). Thus, an individual playing strategy i receives a payoff of 1 when paired with an individual playing strategy $v(i)$ or i, and a payoff of 0 when matched with an individual playing $e(i)$. Individuals reproduce asexually according to their payoffs and every offspring inherits the strategy of its parent.

This description is formalized as continuous-time replicator dynamics:

$$\dot{x}_i = x_i\left(1 - x_{e(i)}\right) - x_i \sum_{j\in\{R,P,S\}} x_j\left(1 - x_{e(j)}\right), \qquad [1]$$

for $i \in \{R, P, S\}$. There is an internal equilibrium at $(\hat{x}_R, \hat{x}_P, \hat{x}_S) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, which is unstable (the real part of the leading eigenvalue is positive). Thus, the three types of players cannot coexist, and one player always dominates (see Figure 1).

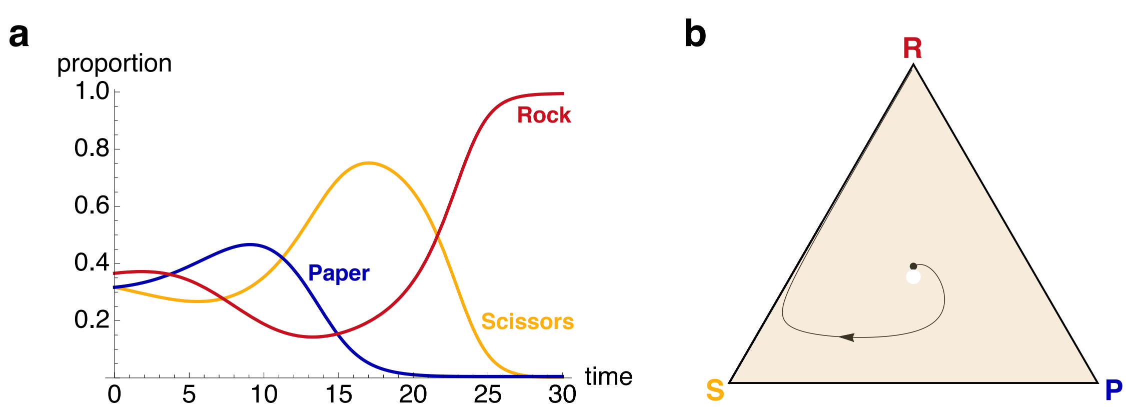

Figure 1: Unstable cycles in the Rock-Paper-Scissors game. (**a**) Using the system governed by equations [1], the strategies undergo cycles of increasing amplitude until one strategy (in this case, Rock) dominates. (**b**) The triangular simplex represents the proportions of each strategy; the closer to each labeled corner, the more common the type at that corner. The white circle represents the (unstable) internal equilibrium. The smaller gray circle gives the initial conditions from part a. The dynamics from part a are given by the trajectory (the arrow indicates direction of "flow" in the simplex).

If we instead assume that the payoffs are 2 for winning, 1 for drawing and 0 for losing, then the dynamics are described by:

$$\dot{x}_i = x_i(2x_{v(i)} + x_i) - x_i \sum_{j\in\{R,P,S\}} x_j(2x_{v(j)} + x_j). \qquad [2]$$

Equations [2] can be rewritten as

$$\dot{x}_i = x_i(x_{v(i)} - x_{e(i)}), \qquad [3]$$

which is the form found in Frean & Abraham (2001), if the rates of replacement are all taken to be unity. In this version of the game, the dynamics involve neutrally stable cycles (see Figure 2).

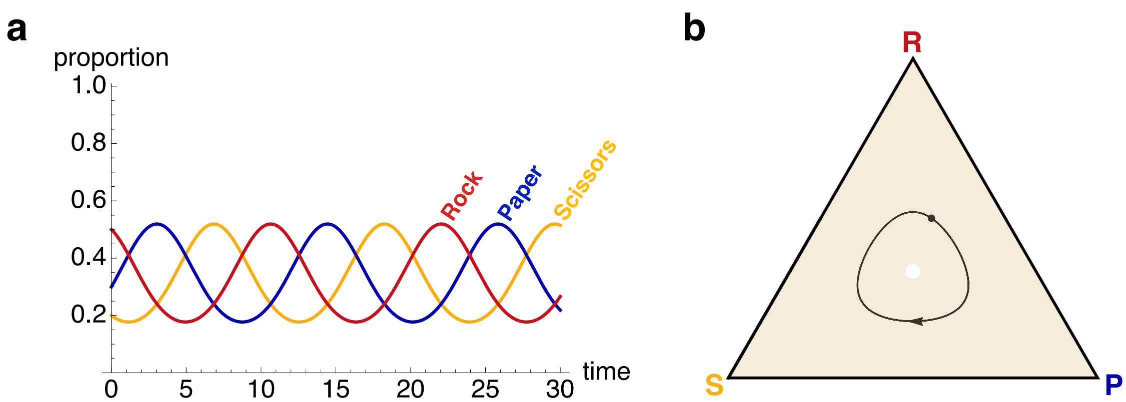

Figure 2: **Neutral cycles in the Rock-Paper-Scissors game**. (**a**) Using the system governed by equations [2], the strategies undergo cycles of constant amplitude. (**b**) The dynamics from part a are given by the trajectory (with the arrow indicating direction of "flow" in the simplex).

The RPS game with bystanders

We reconsider the first version of our game (payoffs are 1 for winning or drawing, and 0 for losing), but with a wrinkle. For each pair that is formed, n bystanders are choosen randomly and independently from the population. If one or more of these bystanders is the enemy of a would-be winner, the game ends in a draw, as the would-be winner is scared off. Because the enemy of a would-be winner is the victim of a would-be loser, the probability that an individual playing strategy i will receive a payoff of 1 is $(1 - x_{e(i)}(1 - x_{v(i)})^n)$. The dynamics are governed by the following equations:

$$\dot{x}_i = x_i(1 - x_{e(i)}(1 - x_{v(i)})^n) - x_i \sum_{j\in\{R,P,S\}} x_j(1 - x_{e(j)}(1 - x_{v(j)})^n). \qquad [4]$$

With only one or two bystanders ($n = 1$ or $n = 2$), the strategies undergo unstable cycles until one strategy dominates. The internal equilibrium is unstable for $n < 2$ (we note that the real part of the leading eigenvalue of the system is zero for $n = 2$). For greater than two bystanders, there is a set of initial conditions where the strategies undergo stable cycles into the internal equilibrium. On the simplex, this is a "basin of attraction" for the locally stable internal equilibrium. The set of initial conditions that allow stable coexistence of strategies (the area of the basin of attraction) grows with the number of bystanders (see Figure 3).

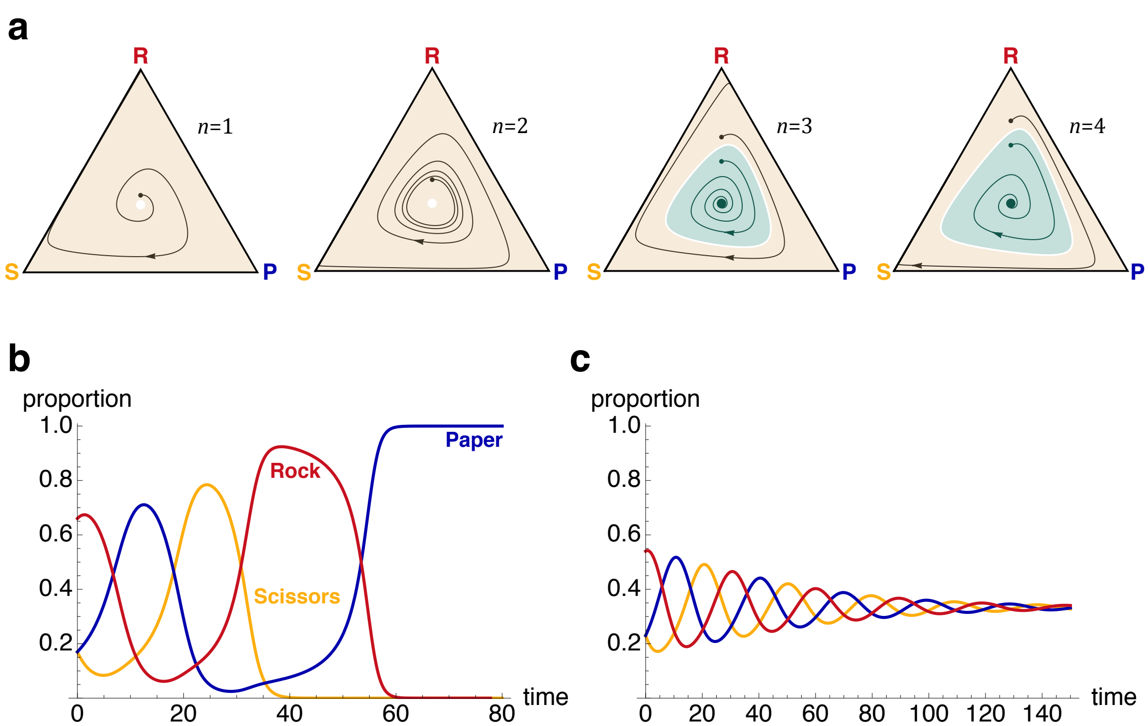

Figure 3: **Dynamics in the Rock-Paper-Scissors game with bystanders**. (**a**) Using the system governed by equations [4], the trajectories show that for one or two bystanders ($n = 1$ or $n = 2$), the strategies undergo unstable cycles. However, for three bystanders ($n = 3$) any population starting in the green basin of attraction undergoes stable cycles into the internal equilibrium. In such cases, stable coexistence follows. Outside the basin, unstable cycles occur. As the number of bystander grows further ($n = 4$ is shown) the area of the basin of attraction grows. The proportion of each strategy is shown for the case of three bystanders ($n = 3$) from part a, where the population starts outside (**b**) or inside (**c**) the basin of attraction.