In what follows, we provide derivations for some of the conditions in the chapter *The Evolution of Altruism* by Glenney & Kerr.

Derivation of condition [10]

Before the first round of reproduction, let the frequency of groups with i altruists be g_i:

$$g_i = \binom{n*}{i} \bar{p}^i (1-\bar{p})^{n*-i}, \qquad \text{[D1]}$$

where $\bar{p}$ is the global frequency of altruists and $n*$ is the group size. After the first round of reproduction, let the frequency of groups with j altruists be given by f_j:

$$f_j = \sum_{i=0}^{n*} \binom{n}{j} \left(\frac{i}{n*}\right)^j \left(\frac{n*-i}{n*}\right)^{n-j} g_i\ , \qquad \text{[D2]}$$

where n is the group size. Assuming a linear fitness structure (equations [6] and [7]), the average fitness of an altruist is:

$$\bar{\alpha} = \frac{1}{n\bar{p}} \sum_{j=0}^{n} f_j\, j(z + b(j-1) - c)\,.$$

The average fitness of a defector is:

$$\bar{\delta} = \frac{1}{n(1-\bar{p})} \sum_{j=0}^{n} f_j (n-j)(z+bj)\,.$$

An increase in altruist frequency requires (by condition [13]):

$$\bar{\alpha} > \bar{\delta}\,,$$

or

$$\sum_{j=0}^{n} f_j \left\{ \frac{j(z+b(j-1)-c)}{n\bar{p}} - \frac{(n-j)(z+bj)}{n(1-\bar{p})} \right\} > 0\,,$$

$$\sum_{j=0}^{n} f_j \left\{ j(z+b(j-1)-c)(1-\bar{p}) - (n-j)(z+bj)\bar{p} \right\} > 0\,,$$

$$\sum_{j=0}^{n} f_j \left\{\left[zj(1-\overline{p})-(n-j)z\overline{p}\right]+\left[bj(j-1)(1-\overline{p})-(n-j)bj\overline{p}\right]-cj(1-\overline{p})\right\}>0\,,$$

$$\sum_{j=0}^{n} f_j \left\{\left[zj-nz\overline{p}\right]+\left[(bj^2-bj)(1-\overline{p})-nbj\overline{p}+bj^2\overline{p}\right]-cj(1-\overline{p})\right\}>0\,,$$

$$\sum_{j=0}^{n} f_j \left\{\left[zj-nz\overline{p}\right]+\left[bj^2-bj+bj\overline{p}-nbj\overline{p}\right]-cj(1-\overline{p})\right\}>0\,,$$

$$\sum_{j=0}^{n} f_j \left\{zj-nz\overline{p}+bj^2-b(1+(n-1)\overline{p})\,j-c(1-\overline{p})\,j\right\}>0\,,$$

$$\sum_{j=0}^{n} f_j \left\{\left[z-b(1+(n-1)\overline{p})-c(1-\overline{p})\right]j+bj^2-nz\overline{p}\right\}>0\,.$$

Using equation [D2], this condition becomes:

$$\sum_{j=0}^{n}\sum_{i=0}^{n^*}\binom{n}{j}\left(\frac{i}{n^*}\right)^{j}\left(\frac{n^*-i}{n^*}\right)^{n-j} g_i \left\{\left[z-b(1+(n-1)\overline{p})-c(1-\overline{p})\right]j+bj^2-nz\overline{p}\right\}>0\,.$$

Using equation [D1], this condition becomes:

$$\sum_{j=0}^{n}\sum_{i=0}^{n^*}\binom{n}{j}\left(\frac{i}{n^*}\right)^{j}\left(\frac{n^*-i}{n^*}\right)^{n-j}\binom{n^*}{i}\overline{p}^{\,i}(1-\overline{p})^{n^*-i}\left\{\left[z-b(1+(n-1)\overline{p})-c(1-\overline{p})\right]j+bj^2-nz\overline{p}\right\}>0\,,$$

or

$$\sum_{i=0}^{n^*}\binom{n^*}{i}\overline{p}^{\,i}(1-\overline{p})^{n^*-i}\sum_{j=0}^{n}\binom{n}{j}\left(\frac{i}{n^*}\right)^{j}\left(\frac{n^*-i}{n^*}\right)^{n-j}\left\{\left[z-b(1+(n-1)\overline{p})-c(1-\overline{p})\right]j+bj^2-nz\overline{p}\right\}>0\,,$$

$$\sum_{i=0}^{n^*}\binom{n^*}{i}\overline{p}^{\,i}(1-\overline{p})^{n^*-i}\left[\begin{array}{l}\left(z-b(1+(n-1)\overline{p})-c(1-\overline{p})\right)\left\{\displaystyle\sum_{j=0}^{n}\binom{n}{j}\left(\frac{i}{n^*}\right)^{j}\left(\frac{n^*-i}{n^*}\right)^{n-j} j\right\}\\ +b\left\{\displaystyle\sum_{j=0}^{n}\binom{n}{j}\left(\frac{i}{n^*}\right)^{j}\left(\frac{n^*-i}{n^*}\right)^{n-j} j^2\right\}-nz\overline{p}\end{array}\right]>0\,. \qquad \text{[D3]}$$

To simplify [D3], we first simplify some of the sums in brackets. Consider the sum:

$$\sum_{j=0}^{n}\binom{n}{j}\left(\frac{i}{n^*}\right)^{j}\left(\frac{n^*-i}{n^*}\right)^{n-j} j\,.$$

This can be reworked as follows:

$$\sum_{j=1}^{n}\frac{n!}{j!(n-j)!}\left(\frac{i}{n^*}\right)^{j}\left(\frac{n^*-i}{n^*}\right)^{n-j} j\,,$$

$$n\left(\frac{i}{n^*}\right)\sum_{j=1}^{n}\frac{(n-1)!}{(j-1)!((n-1)-(j-1))!}\left(\frac{i}{n^*}\right)^{j-1}\left(\frac{n^*-i}{n^*}\right)^{(n-1)-(j-1)}\,.$$

Rewriting the index of the sum as $k = j-1$, we have:

$$n\left(\frac{i}{n^*}\right)\sum_{k=0}^{n-1}\frac{(n-1)!}{k!((n-1)-k)!}\left(\frac{i}{n^*}\right)^{k}\left(\frac{n^*-i}{n^*}\right)^{(n-1)-k}\,,$$

However, the sum is simply all the terms of a binomial distribution (with parameters $n-1$ and i/n^*). Thus, this sum is simply unity, such that:

$$\sum_{j=1}^{n}\frac{n!}{j!(n-j)!}\left(\frac{i}{n^*}\right)^{j}\left(\frac{n^*-i}{n^*}\right)^{n-j} j = n\left(\frac{i}{n^*}\right). \qquad \text{[D4]}$$

This is simply the expectation of a binomially distributed variable (with parameters n and (i/n^*)). Now consider the sum:

$$\sum_{j=0}^{n}\binom{n}{j}\left(\frac{i}{n^*}\right)^{j}\left(\frac{n^*-i}{n^*}\right)^{n-j} j^2\,.$$

This can be reworked as follows:

$$\sum_{j=1}^{n}\frac{n!}{j!(n-j)!}\left(\frac{i}{n^*}\right)^{j}\left(\frac{n^*-i}{n^*}\right)^{n-j} j^2\,,$$

$$n\left(\frac{i}{n^*}\right)\sum_{j=1}^{n}\frac{(n-1)!}{(j-1)!((n-1)-(j-1))!}\left(\frac{i}{n^*}\right)^{j-1}\left(\frac{n^*-i}{n^*}\right)^{(n-1)-(j-1)} j\,,$$

$$n\left(\frac{i}{n^*}\right)\sum_{k=0}^{n-1}\frac{(n-1)!}{k!((n-1)-k)!}\left(\frac{i}{n^*}\right)^{k}\left(\frac{n^*-i}{n^*}\right)^{(n-1)-k}(k+1)\,,$$

$$n\left(\frac{i}{n^*}\right)\left\{\left[\sum_{k=0}^{n-1}\frac{(n-1)!}{k!((n-1)-k)!}\left(\frac{i}{n^*}\right)^{k}\left(\frac{n^*-i}{n^*}\right)^{(n-1)-k} k\right]+\left[\sum_{k=0}^{n-1}\frac{(n-1)!}{k!((n-1)-k)!}\left(\frac{i}{n^*}\right)^{k}\left(\frac{n^*-i}{n^*}\right)^{(n-1)-k}\right]\right\}.$$

Using equation [D4], we can simplify the above quantity:

$$n\left(\frac{i}{n*}\right)\left\{\left[(n-1)\left(\frac{i}{n*}\right)\right]+1\right\}.$$

Thus,

$$\sum_{j=0}^{n}\binom{n}{j}\left(\frac{i}{n*}\right)^{j}\left(\frac{n*-i}{n*}\right)^{n-j}j^{2}=\frac{n(n-1)i^{2}+n*ni}{(n*)^{2}}. \qquad \text{[D5]}$$

We note that equations [D4] and [D5] can be derived much more quickly using a moment generating function approach. Using equations [D4] and [D5], we can simplify equation [D3]:

$$\sum_{i=0}^{n*}\binom{n*}{i}\bar{p}^{i}(1-\bar{p})^{n*-i}\left[\left(z-b(1+(n-1)\bar{p})-c(1-\bar{p})\right)\left\{n\left(\frac{i}{n*}\right)\right\}+b\left\{\frac{n(n-1)i^{2}+n*ni}{(n*)^{2}}\right\}-nz\bar{p}\right]>0,$$

or

$$\sum_{i=0}^{n*}\binom{n*}{i}\bar{p}^{i}(1-\bar{p})^{n*-i}\left[\left\{\left(z-b(1+(n-1)\bar{p})-c(1-\bar{p})\right)+b\right\}\left(\frac{n}{n*}\right)i+b\left\{\frac{n(n-1)}{(n*)^{2}}\right\}i^{2}-nz\bar{p}\right]>0,$$

$$\begin{bmatrix}\left\{\left(z-b(1+(n-1)\bar{p})-c(1-\bar{p})\right)+b\right\}\left(\frac{n}{n*}\right)\left\{\sum_{i=0}^{n*}\binom{n*}{i}\bar{p}^{i}(1-\bar{p})^{n*-i}i\right\}\\+b\left(\frac{n(n-1)}{(n*)^{2}}\right)\left\{\sum_{i=0}^{n*}\binom{n*}{i}\bar{p}^{i}(1-\bar{p})^{n*-i}i^{2}\right\}-nz\bar{p}\end{bmatrix}>0, \qquad \text{[D6]}$$

Using simplifications nearly identical to those used to produce equations [D4] and [D5], we can simplify [D6]:

$$\begin{bmatrix}\left\{\left(z-b(1+(n-1)\bar{p})-c(1-\bar{p})\right)+b\right\}\left(\frac{n}{n*}\right)\left\{n*\bar{p}\right\}\\+b\left(\frac{n(n-1)}{(n*)^{2}}\right)\left\{n*\bar{p}\left[(n*-1)\bar{p}+1\right]\right\}-nz\bar{p}\end{bmatrix}>0,$$

or

$$\left\{-b(n-1)\bar{p}-c(1-\bar{p})\right\}n\bar{p}+b\left(\frac{n(n-1)}{n*}\right)\left\{(n*-1)\bar{p}^{2}+\bar{p}\right\}>0,$$

$$-bn(n-1)\bar{p}^{2}-c(1-\bar{p})n\bar{p}+bn(n-1)\left(\frac{(n*-1)}{n*}\right)\bar{p}^{2}+bn(n-1)\left(\frac{1}{n*}\right)\bar{p}>0,$$

$$-c(1-\overline{p})n\overline{p}-bn(n-1)\left(\frac{1}{n*}\right)\overline{p}^2+bn(n-1)\left(\frac{1}{n*}\right)\overline{p}>0,$$

$$bn(n-1)\left(\frac{1}{n*}\right)\overline{p}(1-\overline{p})>c(1-\overline{p})n\overline{p},$$

$$b(n-1)\left(\frac{1}{n*}\right)>c.$$

With B=b(n–1), the condition for the evolution of altruism is:

$$B\left(\frac{1}{n*}\right)>c,$$

which completes the derivation.

Derivation of condition [12]

Here we consider the case where homogeneous groups of size n form with probability F and groups randomly form with probability 1–F. Given these assumptions, the average fitness of an altruist and defector are:

$$\overline{\alpha}=F(z+b(n-1)-c)+(1-F)\frac{1}{n\overline{p}}\sum_{j=0}^{n}\binom{n}{j}\overline{p}^{j}(1-\overline{p})^{n-j}(z+b(j-1)-c)j,$$

$$\overline{\delta}=Fz+(1-F)\frac{1}{n(1-\overline{p})}\sum_{j=0}^{n}\binom{n}{j}\overline{p}^{j}(1-\overline{p})^{n-j}(z+bj)(n-j).$$

The average altruist fitness can be reworked as follows:

$$\overline{\alpha}=F(z+b(n-1)-c)+(1-F)\frac{1}{n\overline{p}}\sum_{j=0}^{n}\binom{n}{j}\overline{p}^{j}(1-\overline{p})^{n-j}((z-b-c)j+bj^2),$$

$$\overline{\alpha}=F(z+b(n-1)-c)+(1-F)\frac{(z-b-c)}{n\overline{p}}\left\{\sum_{j=0}^{n}\binom{n}{j}\overline{p}^{j}(1-\overline{p})^{n-j}j\right\}+(1-F)\frac{b}{n\overline{p}}\left\{\sum_{j=0}^{n}\binom{n}{j}\overline{p}^{j}(1-\overline{p})^{n-j}j^2\right\}.$$

Using simplifications nearly identical to those used to produce equations [D4] and [D5], we have

$$\overline{\alpha}=F(z+b(n-1)-c)+(1-F)\frac{(z-b-c)}{n\overline{p}}\{n\overline{p}\}+(1-F)\frac{b}{n\overline{p}}\{n\overline{p}[(n-1)\overline{p}+1]\},$$

$$\overline{\alpha}=F(z+b(n-1)-c)+(1-F)(z-b-c)+(1-F)b[(n-1)\overline{p}+1],$$

$$\overline{\alpha} = z - c + F(b(n-1)) + (1-F)b(n-1)\overline{p}\,,$$

$$\overline{\alpha} = z + \left\{F + (1-F)\overline{p}\right\}(b(n-1)) - c\,.$$

The average defector fitness can be reworked as follows:

$$\overline{\delta} = Fz + (1-F)\frac{1}{n(1-\overline{p})}\sum_{j=0}^{n}\binom{n}{j}\overline{p}^{\,j}(1-\overline{p})^{n-j}(z+bj)(n-j)\,,$$

$$\overline{\delta} = Fz + (1-F)\frac{1}{n(1-\overline{p})}\sum_{j=0}^{n}\binom{n}{j}\overline{p}^{\,j}(1-\overline{p})^{n-j}(zn+(bn-z)j-bj^2)\,,$$

$$\overline{\delta} = Fz + (1-F)\frac{z}{(1-\overline{p})} + (1-F)\frac{bn-z}{n(1-\overline{p})}\left\{\sum_{j=0}^{n}\binom{n}{j}\overline{p}^{\,j}(1-\overline{p})^{n-j}j\right\} - (1-F)\frac{b}{n(1-\overline{p})}\left\{\sum_{j=0}^{n}\binom{n}{j}\overline{p}^{\,j}(1-\overline{p})^{n-j}j^2\right\}.$$

$$\overline{\delta} = Fz + (1-F)\frac{z}{(1-\overline{p})} + (1-F)\frac{bn-z}{n(1-\overline{p})}\left\{n\overline{p}\right\} - (1-F)\frac{b}{n(1-\overline{p})}\left\{n\overline{p}\left[(n-1)\overline{p}+1\right]\right\},$$

$$\overline{\delta} = Fz + (1-F)\frac{z}{(1-\overline{p})} + (1-F)\frac{(bn-z)\overline{p}}{(1-\overline{p})} - (1-F)\frac{(n-1)b\overline{p}^2}{(1-\overline{p})} - (1-F)\frac{b\overline{p}}{(1-\overline{p})}\,,$$

$$\overline{\delta} = Fz + (1-F)\left\{\frac{z(1-\overline{p}) + b(n-1)\overline{p} - (n-1)b\overline{p}^2}{(1-\overline{p})}\right\},$$

$$\overline{\delta} = Fz + (1-F)\left\{\frac{z(1-\overline{p}) + b(n-1)\overline{p}(1-\overline{p})}{(1-\overline{p})}\right\},$$

$$\overline{\delta} = z + (1-F)\overline{p}(b(n-1))\,.$$

Condition [13] amounts to:

$$z + \left\{F + (1-F)\overline{p}\right\}(b(n-1)) - c > z + (1-F)\overline{p}(b(n-1))\,,$$

$$F(b(n-1)) - c > 0\,,$$

$$(b(n-1))F > c\,.$$

With $B=b(n-1)$, the condition for the evolution of altruism is:

$$BF > c\,,$$

which completes the derivation.

Derivation of condition [14]

Here we make no assumptions about the way groups form. Thus, we will let f_i be the frequency of groups with i altruists. In this general case, condition [13] is

$$\frac{1}{n\overline{p}}\sum_{i=0}^{n} f_i\alpha_{i-1}i > \frac{1}{n(1-\overline{p})}\sum_{i=0}^{n} f_i\delta_i(n-i)\,,$$

$$(1-\overline{p})\sum_{i=0}^{n} f_i\alpha_{i-1}i > \overline{p}\sum_{i=0}^{n} f_i\delta_i(n-i)\,,$$

$$\sum_{i=0}^{n} f_i\alpha_{i-1}i - \overline{p}\sum_{i=0}^{n} f_i\left\{\alpha_{i-1}i + \delta_i(n-i)\right\} > 0\,. \qquad \text{[D7]}$$

In order to switch to a collective perspective, we note that:

$$\alpha_{i-1}i = \pi_i\phi_i\,,$$

$$\alpha_{i-1}i + \delta_i(n-i) = \pi_i\,.$$

Thus, condition [D7] can be rewritten:

$$\sum_{i=0}^{n} f_i\pi_i\phi_i - \overline{p}\sum_{i=0}^{n} f_i\pi_i > 0\,.$$

$$\sum_{i=0}^{n} f_i\left(\pi_i\phi_i - \pi_i\overline{p}\right) > 0\,,$$

$$\sum_{i=0}^{n} f_i\left(\pi_i p_i - \pi_i\overline{p} + \pi_i\phi_i - \pi_i p_i\right) > 0\,,$$

$$\sum_{i=0}^{n} f_i\left(\pi_i(p_i - \overline{p})\right) + \sum_{i=0}^{n} f_i\pi_i\left(\phi_i - p_i\right) > 0\,, \qquad \text{[D8]}$$

$$\sum_{i=0}^{n} f_i\left((\pi_i - \overline{\pi})(p_i - \overline{p})\right) + \sum_{i=0}^{n} f_i\pi_i\left(\phi_i - p_i\right) > 0\,.$$

Thus, by standard statistical definitions, this condition becomes:

$$\text{cov}(\pi, p) + \text{ave}[\pi(\phi - p)] > 0\,,$$

which completes the derivation.

Derivation of condition [15]

We start by plugging equations [8] and [9] into condition [D8]:

$$\sum_{i=0}^{n} f_i \left((nz+[B-c]i)(p_i-\overline{p})\right)+\sum_{i=0}^{n} f_i \left([z-c+b(i-1)]i-(nz+[B-c]i)\, p_i\right)>0\,.$$

Because $p_i=i/n$ and $B=(n-1)b$, we have:

$$\sum_{i=0}^{n} f_i \left((z+[B-c]p_i)(p_i-\overline{p})\right)+\sum_{i=0}^{n} f_i \left([z-c+b(i-1)]p_i-(z+[(n-1)b-c]p_i)\, p_i\right)>0\,,$$

$$\sum_{i=0}^{n} f_i [B-c](p_i^{\,2}-p_i\overline{p})+\sum_{i=0}^{n} f_i \left([bi-(b+c)]p_i-[(n-1)b-c]p_i^{\,2}\right)>0\,,$$

$$\sum_{i=0}^{n} f_i [B-c](p_i^{\,2}-p_i\overline{p})>\sum_{i=0}^{n} f_i \left([(b+c)-nbp_i]p_i+[(n-1)b-c]p_i^{\,2}\right),$$

$$\sum_{i=0}^{n} f_i [B-c](p_i^{\,2}-p_i\overline{p})>\sum_{i=0}^{n} f_i \left((b+c)\,p_i-nbp_i^{\,2}+(n-1)bp_i^{\,2}-cp_i^{\,2}\right),$$

$$\sum_{i=0}^{n} f_i [B-c](p_i^{\,2}-p_i\overline{p})>\sum_{i=0}^{n} f_i \left((b+c)\,p_i-(b+c)\,p_i^{\,2}\right),$$

$$(B-c)\sum_{i=0}^{n} f_i (p_i^{\,2}-p_i\overline{p})>(b+c)\sum_{i=0}^{n} f_i \left(p_i-p_i^{\,2}\right),$$

$$(B-c)\sum_{i=0}^{n} f_i (p_i^{\,2}-p_i\overline{p})>(b+c)\sum_{i=0}^{n} f_i \left(p_i(1-\overline{p})-(p_i^{\,2}-p_i\overline{p})\right). \qquad \text{[D9]}$$

The variance of altruistic frequency within groups is:

$$\sigma_p^2=\left\{\sum_{i=0}^{n} f_i\, p_i^{\,2}\right\}-\overline{p}^2=\sum_{i=0}^{n} f_i (p_i^{\,2}-p_i\overline{p})\,.$$

Therefore, condition [D9] can be rephrased as follows:

$$(B-c)\sigma_p^2>(b+c)(\overline{p}(1-\overline{p})-\sigma_p^2)\,,$$

which completes the derivation.